

INVARIANT CR MAPPINGS

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Dedicated to Linda Rothschild

1. INTRODUCTION

The subject of CR Geometry interacts with nearly all of mathematics. See [BER] for an extensive discussion of many aspects of CR manifolds and mappings between them. One aspect of the subject not covered in [BER] concerns CR mappings invariant under groups. The purpose of this paper is to discuss interactions with number theory and combinatorics that arise from the seemingly simple setting of group-invariant CR mappings from the unit sphere to a hyperquadric. Some elementary representation theory also arises.

The unit sphere S^{2n-1} in complex Euclidean space \mathbf{C}^n is the basic example of a CR manifold of hypersurface type. More generally we consider the hyperquadric $Q(a, b)$ defined to be the subset of \mathbf{C}^{a+b} defined by

$$\sum_{j=1}^a |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1. \quad (1)$$

Of course S^{2n-1} is invariant under the unitary group $U(n)$. Let Γ be a finite subgroup of $U(n)$. Assume that $f : \mathbf{C}^n \rightarrow \mathbf{C}^N$ is a rational mapping invariant under Γ and that $f(S^{2n-1}) \subset S^{2N-1}$. For most Γ such an f must be a constant. In other words, for most Γ , there is no non-constant Γ -invariant rational mapping from sphere to sphere for any target dimension. In fact, for such a non-constant invariant map to exist, Γ must be cyclic and represented in a rather restricted fashion. See [Li1], [DL], and especially Corollary 7 on page 187 of [D1], for precise statements and the considerable details required.

The restriction to rational mappings is natural; for $n \geq 2$ Forstneric [F1] proved that a proper mapping between balls, with sufficiently many continuous derivatives at the boundary, must be a rational mapping. On the other hand, if one makes no regularity assumption at all on the map, then (see [Li2]) one can create group-invariant proper mappings between balls for any fixed-point free finite unitary group. The restrictions on the group arise from CR Geometry and the smoothness of the CR mappings considered. In this paper we naturally restrict our considerations to the class of rational mappings. See [F2] for considerable discussion about proper holomorphic mappings and CR Geometry.

In order to find group-invariant CR mappings from a sphere, we relax the assumption that the target manifold be a sphere, and instead allow it to be a hyperquadric. We can then always find polynomial examples, as we note in Corollary 1.1. In this paper we give many examples of invariant mappings from spheres to hyperquadrics. Our techniques allow us to give some explicit surprising examples.

In Theorem 6.1 for example, we show that *rigidity* fails for mappings between hyperquadrics; we find non-linear polynomial mappings between hyperquadrics with the same number of negative eigenvalues in the defining equations of the domain and target hyperquadrics. As in the well-known case of maps between spheres, we must allow sufficiently many positive eigenvalues in the target for such maps to exist.

To get started we recall that a polynomial $R : \mathbf{C}^n \times \mathbf{C}^n \rightarrow \mathbf{C}$ is called Hermitian symmetric if $R(z, \bar{w}) = \overline{R(w, \bar{z})}$ for all z and w . If R is Hermitian symmetric, then $R(z, \bar{z})$ is evidently real-valued. By polarization, the converse also holds. We note also that a polynomial in $z = (z_1, \dots, z_n)$ and \bar{z} is Hermitian symmetric if and only if its matrix of coefficients is Hermitian symmetric in the sense of linear algebra.

The following result from [D1] shows how to construct group-invariant mappings from spheres to hyperquadrics. Throughout the paper we will give explicit formulas in many cases.

Theorem 1.1. *Let Γ be a finite subgroup of $U(n)$ of order p . Then there is a unique Hermitian symmetric Γ -invariant polynomial $\Phi_\Gamma(z, \bar{w})$ such that the following hold:*

- 1) $\Phi_\Gamma(0, 0) = 0$.
- 2) *The degree of Φ_Γ in z is p .*
- 3) $\Phi_\Gamma(z, \bar{z}) = 1$ *when z is on the unit sphere.*
- 4) $\Phi_\Gamma(\gamma z, \bar{w}) = \Phi_\Gamma(z, \bar{w})$ *for all $\gamma \in \Gamma$.*

Corollary 1.1. *There are holomorphic vector-valued Γ -invariant polynomial mappings F and G such that we can write*

$$\Phi_\Gamma(z, \bar{z}) = \|F(z)\|^2 - \|G(z)\|^2. \quad (2)$$

The polynomial mapping $z \rightarrow (F(z), G(z))$ restricts to a Γ -invariant mapping from S^{2n-1} to the hyperquadric $Q(N_+, N_-)$, where these integers are the numbers of positive and negative eigenvalues of the matrix of coefficients of Φ_Γ .

The results in this paper revolve around how the mapping (F, G) from Corollary 1.1 depends on Γ . We clarify one point at the start; even if we restrict our considerations to cyclic groups, then this mapping changes (surprisingly much) as the representation of the group changes. The interesting things from the points of view of CR Geometry, Number Theory, and Combinatorics all depend in non-trivial ways on the particular representation. Therefore the results should be considered as statements about the particular subgroup $\Gamma \subset U(n)$, rather than as statements about the abstract group G for which $\pi(G) = \Gamma$.

The proof of Theorem 1 leads to the following formula for Φ_Γ .

$$\Phi_\Gamma(z, \bar{w}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, w \rangle). \quad (3)$$

The first three properties from Theorem 1 are evident from (3), and the fourth property is not hard to check. One also needs to verify uniqueness.

The starting point for this paper will therefore be formula (3). We will first consider three different representations of cyclic groups and we note the considerable differences in the corresponding invariant polynomials. We also consider metacyclic groups. We also discuss some interesting asymptotic considerations, as the order of the group tends to infinity. Additional asymptotic results are expected to appear in the doctoral thesis [G] of D. Grundmeier.

An interesting result in this paper is the application in Section 6. In Theorem 6.1 we construct, for each odd $2p+1$ with $p \geq 1$, a polynomial mapping g_p of degree $2p$ such that

$$g_p : Q(2, 2p+1) \rightarrow Q(N(p), 2p+1). \quad (4)$$

These mappings illustrate a failure of *rigidity*; in many contexts restrictions on the eigenvalues of the domain and defining hyperquadrics force maps to be linear. See [BH]. Our new examples show that rigidity does not hold when we keep the number of negative eigenvalues the same, as long as we allow a sufficient increase in the number of positive eigenvalues. On the other hand, by a result in [BH], the additional restriction that the mapping preserves sides of the hyperquadric does then guarantee rigidity. It is quite striking that the construction of the polynomials in Theorem 6.1 relies on the group-theoretic methods in the rest of the paper.

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2. PROPERTIES OF THE INVARIANT POLYNOMIALS

Let Γ be a finite subgroup of the unitary group $U(n)$, and let Φ_Γ be the unique polynomial defined by (3). Our primary interest concerns how this polynomial depends on the particular representation of the group.

We remark at the outset that we will be considering *reducible* representations. A simple example clarifies why. If G is cyclic of order p , then G has the irreducible unitary (one-dimensional) representation Γ as the group of p -th roots of unity. An elementary calculation shows that the invariant polynomial Φ_Γ becomes simply

$$\Phi_\Gamma(z, \bar{w}) = (z\bar{w})^p. \quad (5)$$

On the other hand, there are many ways to represent G as a subgroup of $U(n)$ for $n \geq 2$. We will consider these below; for now we mention one beautiful special case.

Let p and q be positive integers with $1 \leq q \leq p-1$ and let ω be a primitive p -th root of unity. Let $\Gamma(p, q)$ be the cyclic group generated by the diagonal 2-by-2 matrix A with eigenvalues ω and ω^q :

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}. \quad (6)$$

Because A is diagonal, the invariant polynomial $\Phi_{\Gamma(p, q)}(z, \bar{z})$ depends on only $|z_1|^2$ and $|z_2|^2$. If we write $x = |z_1|^2$ and $y = |z_2|^2$, then we obtain a corresponding polynomial $f_{p, q}$ in x and y . This polynomial has integer coefficients; a combinatorial interpretation of these coefficients appears in [LWW]. The crucial idea in [LWW] is the interpretation of Φ_Γ as a circulant determinant; hence permutations arise and careful study of their cycle structure leads to the combinatorial result. Asymptotic information about these integers as p tends to infinity appears in both [LWW] and [D4]; the technique in [D4] gives an analogue of the Szegő limit theorem. In the special case where $q = 2$, these polynomials provide examples of sharp degree estimates for proper monomial mappings between balls. The polynomials $f_{p, 2}$ have many additional beautiful properties. We pause to write down the formula and state an appealing corollary. These polynomials will arise in the proof of Theorem 6.1.

$$f_{p,2}(x, y) = (-1)^{p+1}y^p + \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^p + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^p. \quad (7)$$

Corollary 2.1 (D4). *Let S_p be the sum of the coefficients of $f_{p,2}$. Then the limit as p tends to infinity of $S_p^{\frac{1}{p}}$ equals the golden ratio $\frac{1+\sqrt{5}}{2}$.*

Proof. The sum of the coefficients is $f_{p,2}(1, 1)$, so put $x = y = 1$ in (7). The largest (in absolute value) of the three terms is the middle term. Taking p -th roots and letting p tend to infinity gives the result. \square

See [DKR] for degree estimates and [DLe] for number-theoretic information concerning uniqueness results for degree estimates. The following elegant primality test was proved in [D2].

Theorem 2.1. *For each q , the congruence $f_{p,q}(x, y) \cong x^p + y^p \pmod{p}$ holds if and only if p is prime.*

We make a few comments. When $q = 1$, the polynomial $f_{p,1}$ is simply $(x + y)^p$ and the result is well-known. For other values of q the polynomials are more complicated. When $q = 2$ or when $q = p - 1$ there are explicit formulas for the integer coefficients. For small q recurrences exist but the order of the recurrences grows exponentially with q . See [D2], [D3], [D4] and [G]. There is no known general formula for the integer coefficients. Nonetheless the basic theory enables us to reduce the congruence question to the special case. Note also that the quotient space $L(p, q) = S^3/\Gamma$ is a Lens space. It might be interesting to relate the polynomials $f_{p,q}$ to the differential topology of these spaces.

We return to the general situation and repeat the crucial point; the invariant polynomials depend on the representation in non-trivial and interesting ways, even in the cyclic case. In order to express them we recall ideas that go back to E. Noether. See [S] for considerable discussion. Given a subgroup Γ of the general linear group, Noether proved that the algebra of polynomials invariant under Γ is generated by polynomials of degree at most the order $|\Gamma|$ of Γ . Given a polynomial p we can create an invariant polynomial by averaging p over the group:

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} p \circ \gamma. \quad (8)$$

We find a basis for the algebra of invariant polynomials as follows. We average each monomial z^α of total degree at most $|\Gamma|$ as in (8) to obtain an invariant polynomial; often the result will be the zero polynomial. The nonzero polynomials that result generate the algebra of polynomials invariant under Γ . In particular, the number of polynomials required is bounded above by the dimension of the space of homogeneous polynomials of degree $|\Gamma|$ in n variables. Finally we can express the F and G from (2) in terms of sums and products of these basis elements. The invariant polynomials here are closely related to the Chern orbit polynomials from [S]. The possibility of polarization makes our approach a bit different. It seems a worthwhile project to deepen this connection. Some results in this direction will appear in [G].

3. CYCLIC GROUPS

Let Γ be cyclic of order p . Then the elements of Γ are $I, A, A^2, \dots, A^{p-1}$ for some unitary matrix A . Formula (3) becomes

$$\Phi_\Gamma(z, \bar{w}) = 1 - \prod_{j=0}^{p-1} (1 - \langle A^j z, w \rangle). \quad (9)$$

We begin by considering three different representations of a cyclic group of order six; we give precise formulas for the corresponding invariant polynomials.

Let ω be a primitive sixth-root of unity, and let η be a primitive third-root of unity. We consider three different unitary matrices; each generates a cyclic group of order six.

Example 3.1. Let Γ be the cyclic group of order 6 generated by A , where

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \quad (10.1)$$

The invariant polynomial satisfies the following formula:

$$\Phi_\Gamma(z, \bar{z}) = (|z_1|^2 + |z_2|^2)^6. \quad (10.2)$$

It follows that Φ is the squared norm of the following holomorphic polynomial:

$$f(z) = (z_1^6, \sqrt{6}z_1^5z_2, \sqrt{15}z_1^4z_2^2, \sqrt{20}z_1^3z_2^3, \sqrt{15}z_1^2z_2^4, \sqrt{6}z_1z_2^5, z_2^6). \quad (10.3)$$

The polynomial f restricts to the sphere to define an invariant CR mapping from S^3 to $S^{13} \subset \mathbf{C}^7$.

Example 3.2. Let Γ be the cyclic group of order 6 generated by A , where

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}. \quad (11.1)$$

The invariant polynomial satisfies the following formula:

$$\Phi_\Gamma(z, \bar{z}) = |z_1|^{12} + |z_2|^{12} + 6|z_1|^2|z_2|^2 + 2|z_1|^6|z_2|^6 - 9|z_1|^4|z_2|^4. \quad (11.2)$$

Note that Φ is not a squared norm. Nonetheless we define f as follows:

$$f(z) = (z_1^6, z_2^6, \sqrt{6}z_1z_2, \sqrt{2}z_1^3z_2^3, 3z_1^2z_2^2). \quad (11.3)$$

Then

$$\Phi = |f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2 - |f_5|^2,$$

and the polynomial f restricts to the sphere to define an invariant CR mapping from S^3 to $Q(4, 1) \subset \mathbf{C}^5$.

Example 3.3. Let Γ be the cyclic group of order 6 generated by A , where

$$A = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix}. \quad (12.1)$$

The polynomial Φ can be expressed as follows:

$$\Phi(z, \bar{z}) = (x + y)^3 + (\eta s + \bar{\eta} t)^3 - (x + y)^3(\eta s + \bar{\eta} t)^3 \quad (12.2)$$

where

$$x = |z_1|^2$$

$$y = |z_2|^2$$

$$s = z_2 \bar{z}_1$$

$$t = z_1 \bar{z}_2.$$

After diagonalization this information determines a (holomorphic) polynomial mapping (F, G) such that

$$\Phi_\Gamma = \|F\|^2 - \|G\|^2.$$

It is somewhat complicated to determine the components of F and G . It is natural to use Noether's approach. For this particular representation, considerable computation then yields the following invariant polynomials:

$$z_1^3 + z_2^3 = p$$

$$z_1^2 z_2 + \eta z_1 z_2^2 = q$$

$$z_1^6 + z_2^6 = f$$

$$z_1^3 z_2^3 = \frac{1}{2}(p^2 - f)$$

$$z_1 z_2^5 + \eta^2 z_1^5 z_2$$

$$\frac{1}{2}(z_1^4 z_2^2 + \eta^2 z_1^2 z_2^4) = h$$

In order to write Φ_Γ nicely, we let

$$g = c(z_1 z_2^5 + 3\eta z_1^3 z_2^3 + \eta^2 z_1^5 z_2).$$

Then one can write Φ_Γ , for some $C > 0$ as follows:

$$\phi_\Gamma = |p|^2 + |q|^2 + \frac{1}{2}(|f - z_1^3 z_2^3|^2 - |f + z_1^3 z_2^3|^2) + C(|g - h|^2 - |g + h|^2). \quad (12.3)$$

We conclude that the invariant polynomial determines an invariant CR mapping (F, G) from the unit sphere S^3 to the hyperquadric $Q(4, 2) \subset \mathbf{C}^6$. We have

$$F = \left(p, q, \frac{1}{\sqrt{2}}(f - z_1^3 z_2^3), \sqrt{C}(g - h) \right) \quad (12.4)$$

$$G = \left(\frac{1}{\sqrt{2}}(f + z_1^3 z_2^3), \sqrt{C}(g + h) \right). \quad (12.5)$$

Consider these three examples together. In each case we have a cyclic group of order six, represented as a subgroup of $U(2)$. In each case we found an invariant CR mapping. The image hyperquadrics were $Q(7, 0)$, $Q(4, 1)$, and $Q(4, 2)$. The corresponding invariant mappings had little in common. In the first case, the map was homogeneous; in the second case the map was not homogeneous, although it was a monomial mapping. In the third case we obtained a rather complicated non-monomial map. It should be evident from these examples that the mappings depend in non-trivial ways on the representation.

4. ASYMPTOTIC INFORMATION

In this section we consider three families of cyclic groups, $\Gamma(p, 1)$, $\Gamma(p, 2)$, and $\Gamma(p, p-1)$. For these groups it is possible to compute the invariant polynomials Φ_Γ exactly. In each case, because the group is generated by a diagonal matrix, the invariant polynomial depends on only $x = |z_1|^2$ and $y = |z_2|^2$. We will therefore often write the polynomials as functions of x and y .

For $p = 1$ we have

$$\Phi_\Gamma(z, \bar{z}) = (|z_1|^2 + |z_2|^2)^p = (x + y)^p. \quad (13)$$

It follows that there is an invariant CR mapping to a sphere, namely the hyperquadric $Q(p+1, 0)$. We pause to prove (13) by establishing the corresponding general result in any domain dimension.

Theorem 4.1. *Let Γ be the cyclic group generated by ωI , where I is the identity operator on \mathbf{C}^n and ω is a primitive p -th root of unity. Then $\Phi_\Gamma(z, \bar{z}) = \|z\|^{2p} = \|z^{\otimes p}\|^2$. Thus $\Phi_{\Gamma(p,1)}^{\frac{1}{p}} = \|z\|^2$ and hence it is independent of p .*

Proof. A basis for the invariant polynomials is given by the homogeneous monomials of degree p . By Theorem 1.1 Φ_Γ is of degree p in z and hence of degree $2p$ overall. It must then be homogeneous of total degree $2p$ and it must take the value 1 on the unit sphere; it therefore equals $\|z\|^{2p}$. \square

We return to the case where $n = 2$ where $\|z\|^2 = |z_1|^2 + |z_2|^2 = x + y$. In the more complicated situation arising from $\Gamma(p, q)$, the expression $\Phi_{\Gamma(p,q)}^{\frac{1}{p}}$ is not constant, but its behavior as p tends to infinity is completely analyzed in [D4].

As an illustration we perform this calculation when $q = 2$. By expanding (3) the following formula holds (see [D4] for details and precise formulas for the n_j):

$$f_{p,2}(x, y) = \Phi_{\Gamma(p,2)}(z, \bar{z}) = x^p + (-1)^{p+1}y^p + \sum_j n_j x^{p-2j}y^j. \quad (14.1)$$

Here the n_j are positive integers and the summation index j satisfies $2j \leq p$. The target hyperquadric now depends on whether p is even or odd. When $p = 2r - 1$ is odd, the target hyperquadric is the sphere, namely the hyperquadric $Q(r+1, 0)$. When $p = 2r$ is even, the target hyperquadric is $Q(r+1, 1)$. In any case, using (7) under the condition $x + \sqrt{x^2 + 4y} > 2y$, we obtain

$$(f_{p,2}(x, y))^{\frac{1}{p}} = \frac{x + \sqrt{x^2 + 4y}}{2} (1 + h_p(x, y))^{\frac{1}{p}}, \quad (14.2)$$

where $h_p(x, y)$ tends to zero as p tends to infinity. Note that we recover Corollary 2.1 by setting $x = y = 1$. We summarize this example in the following result. Similar results hold for the $f_{p,q}$ for $q \geq 3$. See [D4].

Proposition 4.1. *For $x + \sqrt{x^2 + 4y} > 2y$, the limit, as p tends to infinity, of the left-hand side of (14.2) exists and equals $\frac{x + \sqrt{x^2 + 4y}}{2}$.*

It is also possible to compute $\Phi_{\Gamma(p,p-1)}$ exactly. After some computation we obtain the following:

$$\Phi_\Gamma(z, \bar{z}) = |z_1|^{2p} + |z_2|^{2p} + \sum_j n_j (|z_1|^2 |z_2|^2)^j = x^p + y^p + \sum_j n_j (xy)^j, \quad (15)$$

where the n_j are integers. They are 0 when $2j > p$, and otherwise non-zero. In this range $n_j > 0$ when j is odd, and $n_j < 0$ when j is even. Explicit formulas for the n_j exist; in fact they are closely related to the coefficients for $f_{p,2}$. See [D3]. To see what is going on, we must consider the four possibilities for p modulo (4).

We illustrate by listing the polynomials of degrees 4, 5, 6, 7. As above we put $|z_1|^2 = x$ and $|z_2|^2 = y$. We obtain:

$$f_{4,3}(x, y) = x^4 + y^4 + 4xy - 2x^2y^2 \quad (16.4)$$

$$f_{5,4}(x, y) = x^5 + y^5 + 5xy - 5x^2y^2 \quad (16.5)$$

$$f_{6,5}(x, y) = x^6 + y^6 + 6xy - 9x^2y^2 + 2x^3y^3 \quad (16.6)$$

$$f_{7,6}(x, y) = x^7 + y^7 + 7xy - 14x^2y^2 + 7x^3y^3. \quad (16.7)$$

For $\Gamma(p, p-1)$ one can show the following. When $p = 4k$ or $p = 4k+1$, we have $k+2$ positive coefficients and k negative coefficients. When $p = 4k+2$ or $p = 4k+3$, we have $k+3$ positive coefficients and k negative coefficients. For $q > 2$ in general one obtains some negative coefficients when expanding $f_{p,q}$, and hence the target must be a (non-spherical) hyperquadric. The paper [LWW] provides a method for determining the sign of the coefficients.

Given a finite subgroup Γ of $U(n)$, the invariant polynomial Φ_Γ is Hermitian symmetric, and hence its underlying matrix of coefficients is Hermitian. We let $N_+(\Gamma)$ denote the number of positive eigenvalues of this matrix, and we let $N_-(\Gamma)$ denote the number of negative eigenvalues. When Γ is cyclic of order p we sometimes write $N_+(p)$ instead of $N_+(\Gamma)$, but the reader should be warned that the numbers N_+ and N_- depend upon Γ and not just p . The ratio $R_p = \frac{N_+(p)}{N_+(p)+N_-(p)}$ is of some interest, but it can be hard to compute. We therefore consider its asymptotic behavior.

For the class of groups considered above Theorem 4.1 holds. It is a special case of a result to appear in the doctoral thesis [G] of Grundmeier, who has found the limit of R_p for many classes of groups (not necessarily cyclic) whose order depends on p . Many different limiting values can occur. Here we state only the following simple version which applies to the three classes under consideration.

Proposition 4.2. *For the three classes of cyclic groups whose invariant polynomials are given by (13), (14), and (15), the limit of R_p as p tends to infinity exists. In the first two cases the limit is 1. When Φ_Γ satisfies (15), the limit is $\frac{1}{2}$.*

Remark 4.1. For the class of groups $\Gamma(p, q)$ the limit L_q of R_p exists and depends on q . If one then lets q tend to infinity, the resulting limit equals $\frac{3}{4}$. Thus the asymptotic result differs from the limit obtained by setting $q = p-1$ at the start. The subtlety of the situation is evident.

5. METACYCLIC GROUPS

Let C_p denote a cyclic group of order p . A group G is called *metacyclic* if there is an exact sequence of the form

$$1 \rightarrow C_p \rightarrow G \rightarrow C_q \rightarrow 1.$$

Such groups are also described in terms of two generators A and B such that $A^p = I$, $B^q = I$, and $AB = B^m A$ for some m . In this section we will consider metacyclic subgroups of $U(2)$ defined as follows. Let ω be a primitive p -th root of unity, and let A be the following element of $U(2)$:

$$\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}. \quad (17)$$

For these metacyclic groups we obtain in (23) a formula for the invariant polynomials in terms of known invariant polynomials for cyclic groups. We write $C(p, p-1)$ for the cyclic subgroup of $U(2)$ generated by A . Its invariant polynomial is

$$\Phi_{C(p, p-1)} = 1 - \prod_{k=0}^{p-1} (1 - \langle A^k z, z \rangle). \quad (18)$$

Now return to the metacyclic group Γ . Each group element of Γ will be of the form $B^j A^k$ for appropriate exponents j, k . Since B is unitary, $B^* = B^{-1}$. We may therefore write

$$\langle B^j A^k z, w \rangle = \langle A^k z, B^{-j} w \rangle. \quad (19)$$

We use (19) in the product defining Φ_Γ to obtain the following formula:

$$\Phi_\Gamma(z, \bar{z}) = 1 - \prod_{k=0}^{p-1} \prod_{j=0}^{q-1} (1 - \langle B^j A^k z, z \rangle) = 1 - \prod_{k=0}^{p-1} \prod_{j=0}^{q-1} (1 - \langle A^k z, B^{-j} z \rangle). \quad (20)$$

Notice that the term

$$\prod_{k=0}^{p-1} (1 - \langle A^k z, B^{-j} z \rangle) \quad (21)$$

can be expressed in terms of the invariant polynomial for the cyclic group $C(p, p-1)$. We have

$$\prod_{k=0}^{p-1} (1 - \langle A^k z, B^{-j} z \rangle) = 1 - \Phi_{C(p, p-1)}(z, B^{-j} z), \quad (22)$$

and hence we obtain

$$\Phi_\Gamma(z, \bar{z}) = 1 - \prod_{j=0}^{q-1} (1 - \Phi_{C(p, p-1)}(z, B^{-j} z)). \quad (23)$$

The invariance of Φ_Γ follows from the definition, but this property is not immediately evident from this polarized formula. The other properties from Theorem 1.1 are evident in this version of the formula. We have $\Phi_\Gamma(0, 0) = 0$. Also, $\Phi_\Gamma(z, \bar{z}) = 1$ on the unit sphere, because of the term when $j = 0$. The degree in z is pq because we have a product of q terms each of degree p .

The simplest examples of metacyclic groups are the dihedral groups. The dihedral group D_p is the group of symmetries of a regular polygon of p sides. The group D_p has order $2p$; it is generated by two elements A and B , which satisfy the relations $A^p = I$, $B^2 = I$, and $AB = BA^{p-1}$. Thus A corresponds to a rotation

and B corresponds to a reflection. We may represent D_p as a subgroup of $U(2)$ by putting

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad (24.1)$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (24.2)$$

Formula (23) for the invariant polynomial simplifies because the product in (23) has only two terms. We obtain the following result, proved earlier in [D2].

Theorem 5.1. *The invariant polynomial for the above representation of D_p satisfies the following formula:*

$$\Phi(z, \bar{z}) = f_{p,p-1}(|z_1|^2, |z_2|^2) + f_{p,p-1}(z_2 \bar{z}_1, z_1 \bar{z}_2) - f_{p,p-1}(|z_1|^2, |z_2|^2) f_{p,p-1}(z_2 \bar{z}_1, z_1 \bar{z}_2). \quad (25)$$

6. AN APPLICATION; FAILURE OF RIGIDITY

In this section we use the group invariant approach to construct the first examples of polynomial mappings of degree $2p$ from $Q(2, 2p+1)$ to $Q(N(p), 2p+1)$. The key point of these examples is that the number of negative eigenvalues is preserved. The mappings illustrate the failure of rigidity in the case where we keep the number of negative eigenvalues the same but we are allowed to increase the number of positive eigenvalues sufficiently. The mappings arise from part of a general theory being developed [DLe2] by the author and J. Lebl. As mentioned in the introduction, the additional assumption that the mapping preserves sides of the hyperquadric does force linearity in this context. [BH]

Theorem 6.1. *Let $2p+1$ be an odd number with $p \geq 1$. There is an integer $N(p)$ and a holomorphic polynomial mapping g_p of degree $2p$ such that*

$$g_p : Q(2, 2p+1) \rightarrow Q(N(p), 2p+1).$$

and g_p maps to no hyperquadric with smaller numbers of positive or negative eigenvalues.

Proof. We begin with the group $\Gamma(2p, 2)$. We expand the formula given in (7) with p replaced by $2p$. The result is a polynomial $f_{2p,2}$ in the two variables x, y with the following properties. First, the coefficients are positive except for the coefficient of y^{2p} which is -1 . Second, we have $f_{2p,2}(x, y) = 1$ on $x + y = 1$. Third, because of the group invariance, only even powers of x arise. We therefore can replace x by $-x$ and obtain a polynomial $f(x, y)$ such that $f(x, y) = 1$ on $-x + y = 1$ and again, all coefficients are positive except for the coefficient of y^{2p} . Next replace y by $Y_1 + Y_2$. We obtain a polynomial in x, Y_1, Y_2 which has precisely $2p+1$ terms with negative coefficients. These terms arise from expanding $-(Y_1 + Y_2)^{2p}$. All other terms have positive coefficients. This polynomial takes the value 1 on the set $-x + Y_1 + Y_2 = 1$. Now replace x by $X_1 + \dots + X_{2p+1}$.

We now have a polynomial $W(X, Y)$ that is 1 on the set given by

$$-\sum_{j=1}^{2p+1} X_j + \sum_{j=1}^2 Y_j = 1.$$

It has precisely $2p + 1$ terms with negative coefficients. There are many terms with positive coefficients; suppose that the number is $N(p)$. In order to get back to the holomorphic setting, we put $X_j = |z_j|^2$ for $1 \leq j \leq 2p + 1$ and we put $Y_1 = |z_{2p+2}|^2$ and $Y_2 = |z_{2p+3}|^2$. We note that this idea (an example of the moment map) has been often used in this paper, as well as in the author's work on proper mappings between balls; see for example [D1] and [DKR]. Let $g_p(z)$ be the mapping, determined up to a diagonal unitary matrix, with

$$\sum_{j=1}^{N(p)} |g_j(z)|^2 - \sum_{j=1}^{2p+1} |g_j(z)|^2 = W(X, Y). \quad (26)$$

Each component of g_p is determined by (26) up to a complex number of modulus 1. The degree of g_p is the same as the degree of w . We obtain all the claimed properties. \square

Example 6.1. We write out everything explicitly when $p = 1$. Let $c = \sqrt{2}$. The proof of Theorem 6.1 yields the polynomial mapping $g : Q(2, 3) \rightarrow Q(8, 3)$ of degree 2 defined by

$$g(z) = (z_1^2, z_2^2, z_3^2, cz_1z_2, cz_1z_3, cz_2z_3, cz_4, cz_5; z_4^2, cz_4z_5, z_5^2). \quad (27)$$

Notice that we used a semi-colon after the first eight terms to highlight that g maps to $Q(8, 3)$. Summing the squared moduli of the first eight terms yields

$$(|z_1|^2 + |z_2|^2 + |z_3|^2)^2 + 2(|z_4|^2 + |z_5|^2). \quad (28)$$

Summing the squared moduli of the last three terms yields

$$(|z_4|^2 + |z_5|^2)^2. \quad (29)$$

The set $Q(2, 3)$ is given by

$$|z_4|^2 + |z_5|^2 - 1 = |z_1|^2 + |z_2|^2 + |z_3|^2.$$

On this set we obtain 1 when we subtract (29) from (28).

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